Arithmetic, pre-algebra, and algebra: a model of transition

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Learning to operate algebraically requires assimilation of new mathematical concepts and procedures. Current literature identified a gap between arithmetic and algebra and proposed a pre-algebra level. This paper reports on a longitudinal study that investigated students' readiness for algebra, from a cognitive perspective, to determine what constitutes a pre-algebraic level of understanding. Thirty-three students in grades 7, 8, and 9 participated. A two-path model depicts the transition from arithmetic to pre-algebra to algebra; students' understanding of relevant knowledge is discussed.

Pre-algebraic and Algebraic Knowledge

Secondary students often seem unable to apply basic algebraic concepts and skills and do not appear to understand many of the underlying structures. This becomes evident when a distinction is drawn between performance and understanding as outcomes of instruction (Rosnick & Clements, 1980). Recent research has focused on the transition from arithmetic to algebra and the difficulties in developing algebraic concepts caused by a cognitive gap (Herscovics & Linchevski, 1994) or didactic cut (Filloy & Rojano, 1989). It is suggested that the gap/cut is located between the knowledge required to solve arithmetic equations, by inverting or undoing, and the knowledge required to solve algebraic equations by operating on or with the unknown or variable. Linchevski and Herscovics (1996) found that students could not operate spontaneously on or with the unknown and that grouping algebraic terms is not a simple problem. They argued that students viewed algebraic expressions intuitively as computational processes (cf. Sfard & Linchevski, 1994) and suggested that in teaching, instead of moving from variable to expression to equation, arithmetical solution of linear equations might be more suitable initially for learning to operate on or with the unknown. Filloy and Rojano (1989) believe such concerns indicate the need for an operational level of 'pre-algebraic knowledge' between arithmetic and algebra.

However descriptions in the literature of what constitutes a pre-algebraic level of understanding are unclear. For example, Herscovics and Linchevski (1994) consider that the cognitive gap defines a level of pre-algebra and regard this as "involving those intuitive algebraic ideas stemming from the presence of an unknown in a first degree equation" (p. 75). Linchevski (1995) provided an explanation for pre-algebra as incorporating substitution of numbers for letters and allowing students to build cognitive schemes through reflective activity and spontaneous procedures. In contrast to this, Bell (1996) proposed six hypotheses about algebraic thought. These included: resolution of complex arithmetic problems by step-by-step methods working from given data to unknowns or by global perceptions and use of multiple arithmetic relations; recognition and use of general properties of the number system and its operations; and use of a manipulable symbolic language to aid this work. We believe that these hypotheses are concerned both with pre-algebraic and algebraic thought and that a sound arithmetic knowledge base is essential to developing pre-algebraic concepts. Research has shown however that students' understanding of arithmetic principles is often inconsistent and can be a source of cognitive difficulties in acquiring such concepts.

Cognitive Difficulties

The Arithmetic Knowledge Base

Algebra has been depicted as emerging from arithmetic and constructed in terms of students' prior knowledge of symbols, operations, and laws which are extended into higher order abstraction. For example, students need to form a more abstract view of addition in algebra as an object in Sfard's sense (see Linchevski, 1995). The function of operations must also be extended and new knowledge assimilated in an algebraic framework. Booker (1987) exemplified this by stating that in order to solve equations, multiplication processes must include factorisation. Gallardo and Rojano (1987) reported difficulties students had in understanding algebraic principles that stemmed from an inadequate arithmetic knowledge base. These included difficulty with inversion of operations, the need to accommodate a move from operating vertically in arithmetic to working with equations presented horizontally in algebra, and the nature of equality. Linchevski and Herscovics (1994) conducted a study with grade six students which also found inadequacies in students' arithmetic knowledge base. They reported that students over-generalised order of operations, failed to perceive cancellation thus resulting in sequential operation of equations, and displayed a static view of the use of brackets. They considered all of these obstacles to be of a pre-algebraic nature.

Booth (1989) stressed the importance of students' understanding various structural notions in arithmetic. Herscovics and Linchevski (1994) examined this contention by analysing the knowledge required to solve $4 + n - 2 + 5 = 11 + 3 - 5$. They stated that students need to be able to use commutativity to obtain $(n + 4) - 2 + 5$ and associativity to perform $[(4 - 2) + 5]$. However MacGregor (1996) suggested that students possess an unsure understanding of the commutative law while Booth (1988) reported students believe that division, like addition, is commutative. A sound understanding of the distributive law is also essential for algebraic functioning (Demana & Leitzel, 1988). Students' conception of equals has been documented (Herscovisc & Linchevski, 1994; Kieran, 1981; Linchevski, 1995) as indicating an operation to be performed on the left of the equal sign with the answer appearing to the right. Linchevski argued from a psychological point of view, that operating algebraically requires students to move from a unidirectional mode of reading an equation to multi-directional processing of information. Clearly, arithmetic principles are multifaceted; if they not well understood they may result in cognitive difficulties for students developing new algebraic skills.

Unknowns and Variables

The move from arithmetic to algebra also requires students' conceptions of operations performed on numbers to change so that the concept of operating on variables may develop (Filloy & Rojano, 1989). Conceptual obstacles in interpreting letters in algebra have included a lack of understanding concatenation. Herscovics and Linchevski (1994) found only 36% of 14 year old students gave a correct answer when asked to add 4 onto $3n$. They also reported that seventh graders made errors when substituting a number for n in $3n$ such as writing 32.

We consider that understanding of the unknown and solving to find the unknown in an equation constitute, in part, a pre-algebraic level of understanding. Panizza, Sadovsky, and Sessa (1997) suggest that the notion of unknown may become an epistemological obstacle when trying to conceptualise the notion of variable. However others, such as Graham and Thomas (1997), maintain that allowing students to gain an appreciation of letters as labelled stores will help develop an understanding that will improve assimilation of later concepts. This notion has significant implications in light of Ursini and Triguero's (1997) finding that college students had difficulty discriminating between variable as unknown and variable as generalised number. They proposed understanding of variable as unknown implies: recognising and identifying in a problem situation the presence of something unknown that can be determined by considering the restrictions of the problem; the ability to substitute for the variable, the value or values that make the equation true; and determining the unknown by performing the required arithmetic and/or algebraic operations.

A Sequence for Learning Complex Equations

Complex linear equations in algebra such as $2x+3=11$ include three crucial components: an equals sign, a series of more than one operation, and a variable 'x'. We are describing these equations as complex, because they include more than one operation, as opposed to binary operations such as $x+5=6$. We propose a two-path model for learning complex algebra where binary arithmetic (2+3=5), complex arithmetic (35÷7+8=13) and complex pre-algebraic operations [$3(x+7)=24$] are necessary components of one path and binary arithmetic (2+3=5), binary pre-algebraic ($x+7=16$) and binary algebraic ($x+y=12$) operations are necessary components of a second path. This means that understanding binary operations, such as $2x$ and $x+3$, should be a prerequisite to understanding $2x+3=11$ as should application of operational laws to series of operations. Additionally, we suggest that equations such as $x+7=16$ require solution procedures of a pre-algebraic nature which, at the lowest level, comprise use of inverse arithmetical procedures to find the unknown. The two-path model also assumes that learning linear algebraic equations will be facilitated by understanding isomorphic structures in complex arithmetic. The developmental literature (Collis, 1974; Sfard & Linchevski, 1994) supports this sequence in that it suggests acquisition of pre-algebraic and algebraic concepts in the following order: one occurrence of the unknown in binary operations, a series of operations on and with numbers and the unknown, multiples of the unknown, acceptance of lack of closure and immediate solution with a series of operations on the unknown, and finally relationships between two variables and operations on them.

The purpose of our study was to explore students' early understandings of algebraic concepts as they moved from arithmetic to algebraic. This was to determine: (a) the validity of the two-path model of sequential development of algebraic understanding and (b) what constituted a pre-algebraic level of understanding. Results from pilot work were published in Boulton-Lewis, et al., (in press), and for the pilot work and the first year of the study in Boulton-Lewis et al. (1997) and Cooper et al. (1997). This paper presents results for the three-years of the longitudinal study.

Method

Sample

The sample comprised 33 students who were tested in the first year, in grade 7, in four state primary schools in Brisbane. These were feeder schools for the high school where, in the second and third years of the study, students were in grade 8 then grade 9. Generally these schools were in a middle socio-economic area. Interviews were conducted with the grade 7 students before any formal algebra instruction took place and with grade 8 students after they had received instruction in operational laws, use of brackets, and solution of arithmetic word and number problems. Grade 9 students were interviewed after they had learned about an 'unknown' in a linear equation and solution of a linear equation using balance procedures.

Tasks and Procedure

Students were presented with expressions and equations written on cards and asked questions that investigated: commutative (x, -, ÷, +; 35 ? 76=76 ? 35) and distributive (6 x 13 = 60 + 18) laws; inverse operations (5x71=355, 355 ? 5 = 71; 64-29=35, 35 ? 29=64); order of operations [32+(12x8)÷3]; meaning of equals in an incomplete equation (28÷7+20=) and a complete equation (28÷7+20=60-36); meaning of unknown (+5=9; $x+7=16$; $3x$ = 12) and variable (+5; $3x$);); and solution of linear equations [$3x+7=22$; $3(x+7)=24$]. Students were interviewed individually and videotaped. They were encouraged to complete each task, however, if they could not the interviewer proceeded to the next task.

Analysis

Interviews were transcribed and analysed to identify key categories. The NUD*IST program (Richards & Richards, 1994) was used to classify response protocols under these categories and further sub-categories. Responses for laws, inverse operations and order of operations were categorised as *satisfactory* or *unsatisfactory* as a basis for learning algebra. Responses for the other tasks were categorised as *inappropriate*, indicating a lack of knowledge required for the task; *arithmetic*, focussing on arithmetical procedures and numerical answers; *pre-algebraic*, evidencing understanding between arithmetic procedures and intuitive algebraic ideas (Herscovics & Linchevski, 1994) and use of inverse procedures; and *algebraic*, evidencing recognition of relationships expressed in simplified form and use of general properties of the number system and its operations (Bell, 1996).

Results

The summary results below will be illustrated by Tables and examples of responses at the conference presentation.

Commutative and Distributive Laws, Inverse Operations and Order of Operations

In grades 7 and 8 the majority of students (19 and 17 respectively) could not explain *commutativity* of addition and multiplication satisfactorily. However by grade 9, 25 students gave a satisfactory explanation for commutativity. The majority of students in grades 7 (21) and 8 (17) could not give a satisfactory explanation for the *distributive* law and while a substantial number of students (14) still could not explain this law in grade 9, 19 students were able to provide a satisfactory explanation. *Inverse operations* were explained satisfactorily by the majority of students in each grade (26, 30, and 33 respectively) and by grades 8 and 9 most students explained *order of operations* satisfactorily (26 and 23 respectively) compared with only nine satisfactory explanations in grade 7.

Meaning of Equals

In each grade, the majority of students explained '=' in $28 \div 7 + 20 =$ as find the answer. Only one response in grade 8 and three responses in grade 9 evidenced knowledge that '=' denoted an equivalence relationship by stating that both sides had to be equal. For '=' in $28\div7+20=60-36$, the majority of responses moved from arithmetic in grade 7 when 19 students stated equals meant the answer, to arithmetic (12) or algebraic (12) in grade 8 as students explained equals as either the answer or denoting equivalence, to algebraic in grade 9 with most students (19) explaining equals as equivalence or showing a balanced equation.

Meaning of Unknown and Variable

The majority of students in each grade indicated that , in $+ 5 = 9$ (16, 22, and 21 respectively), and x in $x + 7 = 16$ (18, 24, and 26 respectively) represented an unknown number. However when x was presented in $3x = 12$ in grade 7 most students (18) did not know what this meant and gave an inappropriate explanation. In grade 8 most students explained concatenated x either arithmetically as a times sign (12) or pre-algebraically as an unknown number (12). In grade 9 most students' (25) explanations for x in $3x = 12$ were pre-algebraic stating that x was an unknown number.

For meaning of variable in grade 7 most students (18) stated pre-algebraically that in $+ 5$ represented an unknown number; another eight students stated algebraically it was any number. Five students gave inappropriate responses and two stated it was the answer. In grades 8 and 9 the majority of students stated pre-algebraically that was an unknown number (15 and 18 respectively) or algebraically that is was any number (14 each grade). In grade 9 there was only one arithmetic response and this indicated was the answer. For x in $3x$ most students in grades 7 (19) and 8 (15) responded arithmetically that it was a times (multiplication) sign. However by grade 9 the majority of students (17) stated pre-algebraically that it was an unknown number and a further 10 students responded algebraically that it represented any number.

Solution of Linear Equations

The majority of students in grades 7 (14) and 8 (13), as one would expect, did not know how to solve $3x + 7 = 22$. Eight students in grade 7 and 10 students in grade 8 used inverse arithmetic processes to find what they believed was missing after x because they interpreted x as a 'times' sign. Nine students in grade 7 and 10 students in grade 8 used inverse processes to solve for x which was categorised as pre-algebraic. By grade 9 most students (23) solved $3x + 7 = 22$ pre-algebraically by using inverse processes. Two students did not know how to solve the equation and two used an incomplete pre-algebraic balance method which entailed balancing the equation by taking 7 from both sides, then dividing 15 by 3, rather than dividing each side by 3. Six students solved by using a complete balance procedure which was categorised as algebraic.

For $3(x + 7) = 24$ the majority of students (27) in grade 7 did not know how to solve the equation, while six students used a pre-algebraic inverse procedure. By grade 8 the majority of students still did not know how to solve the equation, however 12 students did use a pre-algebraic inverse procedure. Six students used arithmetic processes: two were inverse and depended on finding the space after the x and four were trial and error. By grade 9 most students (19) used pre-algebraic inverse processes to solve $3(x + 7) = 24$. Another four responses were pre-algebraic: three were incomplete balance and one was the incorrect balance method. Six responses were inappropriate, one student used a trial and error arithmetic process, and three students used a complete balance procedure which was categorised as algebraic.

Discussion

It was not until grade 9 that most students had sufficient understanding of the commutative and distributive laws to apply these to linear equations. These findings attest to MacGregor (1996) and Booth's (1988) contention that students have difficulty in understanding the commutative law. It also indicates a need for explicit instruction in these laws, in particular as Demana and Leitzel (1988) maintain that a sound understanding of the distributive property is essential for algebraic functioning. The majority of students displayed a satisfactory understanding of inverse procedures and correct order of operations by grade 8. Herscovics and Linchevski (1994) include understanding the order of operations as indicative of arithmetic functioning. Overall these results showed that it was not until grade 9 that most students displayed a satisfactory understanding of these arithmetic principles to apply them to algebra. Even then some students exhibited unsatisfactory understanding of each principle, except inverse operations, and would require instruction to facilitate understanding before they could apply them in an algebraic situation.

For equals in the incomplete equation, the majority of students each year indicated an arithmetic understanding by stating it meant find the answer. For the complete equation, understanding of equals moved from arithmetic in grade 7, to arithmetic or algebraic in grade 8, with most students in grade 9 stating algebraically that '=' denoted an equal or balanced relationship. However in all three grades almost one third of the students interpreted '=' pre algebraically. Kieran (1981) noted that students require an equivalence understanding of equals to operate algebraically. By grade 9, 19 students demonstrated an equivalence understanding, however 14 students were still operating at either a pre-algebraic or arithmetic level. This suggests that while students' knowledge of '=' had developed, there was still a substantial number of students who did not understand '=' in an algebraic sense and would need to learn the concept of equivalence. Providing explicit instruction of equals at a pre-algebraic level, that is each side is the same, may bridge the gap between arithmetic and algebraic understanding of equals.

Most students, over the three years, knew that in the expression and equation represented an unknown number. In the expression this is indicative of a pre-algebraic level of understanding as could be interpreted algebraically as representing 'any number'. In grades 8 and 9, 14 students did explain , in + 5, as any number and as understanding emerged in grade 9 some said it was a variable. These results indicate that understanding initially as an unknown number appears to be a suitable foundation from which to introduce the concept of any number or variable. Similarly by grade 9 most students explained x (not concatenated) in the equations as an unknown number.

Understanding of x in $3x$ was a more cognitively demanding task. Most students in grades 7 and 8 did not understand concatenated x and hence could not solve the linear equations. Allowing students opportunity to experience letters as labelled stores earlier in arithmetic instruction, as Graham and Thomas (1997) suggest, may foster intuitive understanding of concatenation and subsequent algebraic calculations. In grade 9, after students had been instructed in concatenation and use of inverse or balance procedures to solve an equation, most chose to use inverse pre-algebraic procedures to solve linear equations; a smaller number of students used balance procedures successfully. Sfard and Linchevski (1994) view the process of solving to find an unknown by reversing procedures or backtracking, as early algebraic thinking. We suggest that this solution process would be more appropriately placed at a pre-algebraic level of functioning.

Conclusion

Results for the three years of this study support the case for focussing on an explicit pre-algebraic level of understanding. This was particularly evident in students' unsatisfactory explanations of commutative and distributive laws, explanations for equals in the complete equation, and x in the expressions, and solution of the linear equations. The findings also support the sequence of instruction, as proposed in the model, that understanding of binary operations such as $3x$ is a prerequisite to solution of complex algebraic equations. We suggest that as arithmetic procedures are applied intuitively they should be taught explicitly in order to provide a sound basis for pre-algebraic instruction. Our findings indicated certain inadequacies in students' arithmetic knowledge base. In particular we feel there is a need for explicit and prolonged instruction in commutative and distributive laws and that letters could be introduced earlier, in arithmetic, to represent labelled stores. Finally we propose that pre-algebra should include instruction in: operational laws; equals as equality of sides leading to equivalence; solution of binary and complex equations using inverse procedures; use of letters to represent unknowns as distinct from variables; extensive use of 'x' as 'times' between an unknown and its coefficient; concatenation; and should be based on students' arithmetic as well as intuitive algebraic knowledge.

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